Polynomial Functions

Definition - $f(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_0$ **where a's are real coefficients and exponents are 0, 1, 2, 3…**

Can you identify polynomials?

Adding / Subtracting / Multiplying / Dividing Polynomials

Adding / Subtracting – just add like terms – but be careful to distribute (-1) when subtracting!!!

Note: $(x^2+3x-1)-(4x^2-2x)$ is subtraction!!! Not multiplication…

Multiplying – multiply <u>each</u> term by <u>each</u> term, and then add – "super" distribution of terms

Can you multiply $(x-3)(2x+5)(3x-1)$?

Division –

Long Division

$$\begin{array}{r|l}
 & 2x^2 - 3x + 5 \quad \leftarrow \text{quotient} \\
x^2+3x+2 & 2x^4+3x^3+0x^2+5x-1 \\
 & \underline{2x^4+6x^3+4x^2} \qquad \text{Multiply divisor by } \frac{2x^4}{x^2}=2x^2. \\
 & -3x^3-4x^2+5x \qquad \text{Subtract. Bring down next term.} \\
 & \underline{-3x^3-9x^2-6x} \qquad \text{Multiply divisor by } \frac{-3x^3}{x^2}=-3x. \\
 & 5x^2+11x-1 \qquad \text{Subtract. Bring down next term.} \\
 & \underline{5x^2+15x+10} \qquad \text{Multiply divisor by } \frac{5x^2}{x^2}=5. \\
 & -4x-11 \quad \leftarrow \text{remainder}
\end{array}$$

$$\frac{2x^4+3x^3+5x-1}{x^2+3x+2}=2x^2-3x+5+\frac{-4x-11}{x^2+3x+2}$$

Synthetic Division

Know how to find the subsequent quotient after division and further factor this to find other <u>solutions</u>

<u>**Note**</u>: Don't forget to put a 0 for the missing x terms (including the last constant if missing)

Ex. $(-x^3+4x^2+9)\div(x-3)$

$$\begin{array}{r|rrrr}
3 & -1 & 4 & 0 & 9 \\
 & & -3 & 3 & 9 \\
\hline
\text{coefficients of quotient} \rightarrow & -1 & 1 & 3 & 18 \leftarrow \text{remainder}
\end{array}$$

$$\frac{-x^3+4x^2+9}{x-3}=-x^2+x+3+\frac{18}{x-3}$$

KEY CONCEPT: If the remainder is 0 when dividing by polynomial then polynomial is a factor.

Is $(x+5)$ a factor of $3x^4-2x^2+10x-30$ - why?

Special patterns

Know these to expand and also to factor (i.e. – be able to go in either direction)

$(a+b)(a-b)=a^2-b^2$

$(a+b)^2=a^2+2ab+b^2$

$(a-b)^2=a^2-2ab+b^2$ ← **NOTE:** $(x+10)^2\neq x^2+100$ **!!!!! NEVER !!!! PLEASE DON'T DO THIS!!**

$(a+b)(a^2-ab+b^2)=a^3+b^3$

$(a-b)(a^2+ab+b^2)=a^3-b^3$

Pascal's Triangle for binomial expansion

Know how to use Pascal's Triangle to expand $(a+b)^n$

Ex. $(a+b)^6 = 1a^6b^0 + 6a^5b^1 + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6a^1b^5 + 1a^0b^6$

$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Solving Polynomials Equations – FACTOR!!

KEY CONCEPT – We use factoring to help us solve polynomial equations to "break" large exponents down into smaller x^2 or x type of equations that we know how to solve. **Set one side of equation to 0!!!!**

After factoring, we use the **Zero Product Property** to find solutions

Ex.

$$5x^3 + 15x^2 + 12x + 36 = 0$$
$$5x^2(x+3) + 12(x+3) = 0$$
$$(5x^2+12)(x+3) = 0$$
$$5x^2 + 12 = 0 \quad \text{or} \quad x+3=0$$
$$x = \pm\frac{2\sqrt{15}}{5}i \quad \text{or} \quad x=-3$$

Ex.

$$f(x) = \frac{1}{4}x^2(x-3)(x+2)^3(x^2+9)(x^2+2x+5)(5x-7)$$

Find all x such that f(x) = 0

$$x = 0$$
$$x-3=0 \Rightarrow x=3$$
$$x+2=0 \Rightarrow x=-2$$
$$x^2+9=0 \Rightarrow x=\pm 3i$$
$$x^2+2x+5=0 \Rightarrow x=\frac{-2\pm\sqrt{4-20}}{2} = -1\pm 2i$$
$$5x-7=0 \Rightarrow x=\frac{7}{5}$$

Factoring – 4 methods – p. 353

Common factor – always try this first

Special pattern – (see **Special Patterns** above)

Quadratic Form - $x^8 - 5x^4 + 6 = (x^4-2)(x^4-3)$

Grouping (see example above) – be careful when grouping with negatives signs

NOTE: Keep factoring until all done – especially look for special patterns like x^2-9 which can be factored down more!!

NOTE: Use zero information to help factor‼

If you know 2 is a zero, then (x-2) is a factor

If f(2) = 0, then (x-2) is a factor

Rational Root Theorem

Possible roots of $f(x) = a_nx^n + \cdots + a_0$ - all combinations of $\pm\frac{factors\ of\ a_0}{factors\ of\ a_n}$

For each possible root, check if $f(PossibleRoot) = 0$ (using synthetic division or plug-in)

Graphing Polynomials Equations – Factor polynomial to "intercept" form

Step 1: Plot zeros [example: factor (x + 3) has x-intercept at x = -3]

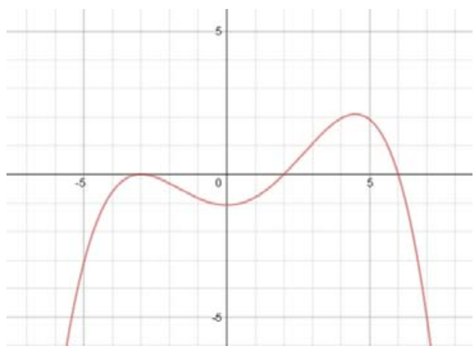
Step 2: Determine end behavior - determine leading coefficient and highest exponent (even/odd)

$+ax^{odd} \rightarrow -\infty, +\infty$ $-ax^{odd} \rightarrow +\infty, -\infty$ <u>NOTE</u>: Like a line $y = mx^1$ ← odd

$+ax^{even} \rightarrow +\infty, +\infty$ $-ax^{even} \rightarrow -\infty, -\infty$ <u>NOTE</u>: Like a parabola $y = ax^2$ ← even

Step 3: Determine crossing/tangent behavior at zeroes

$(x - k)^{odd} \rightarrow$ crossing $(x - k)^{even} \rightarrow$ tangent



$$y = -\frac{1}{100}(x-2)(x+3)^2(x-6)$$

Zeroes: 2, -3, 6

End Behavior: like $-\frac{1}{100}x^4$ so $-\infty, -\infty$

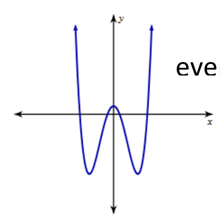
Zero behavior:

Tangent at -3 / Crossing at 2 / Crossing at 6

Turning points – local minumum or maximum where function changes from increasing to decreasing (or vice versa)

In n turning points, then polynomial is at least n+1 degree

If n non-repeating real zeroes, then at least n-1 turning points

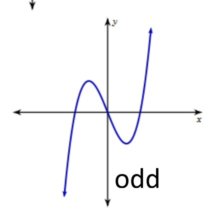


even

Even / Odd functions

Even when $f(-x) = f(x)$ - reflection over y-axis

Odd when $f(-x) = -f(x)$ - rotation 180° around origin



odd

Polynomials –

Even if all terms are even degree – (includes the constant term) $ax^4 + bx^2 + cx^0$

Odd if all the terms are odd degree – ex. $ax^7 + bx^5 + cx^1$

Neither if a mixture of even and degree terms = ex. $ax^6 + bx^3 + 4$

Location Principle

If $f(a) < 0$ and $f(b) > 0$, then there must be at least 1 zero between a and b.

x	y
3	-3
4	-10
5	15

← must be a zero between 4 and 5

Fundamental Theorem of Algebra

There are n solutions of f(x)=0 for a polynomial of degree n. (This includes "repeated" zeroes).

Ex. $f(x) = 3x - 1$ 1 solution - we know this from previous chapters

Ex. $f(x) = 3x^2 - 3x + 1$ 2 solutions - we know this from previous chapters

Ex. $f(x) = 3x^8 - 4x^3 + 2x^2 + 5$ 8 solution because polynomial is of degree 8

Conjugate Theorems

For rational coefficient polynomials, **irrational solutions always come in conjugate pairs.**

For real coefficient polynomials, **complex solutions always come in conjugate pairs.**

Ex. if you know 4 + 3i is a solution, then 4 – 3i is also a solution

Ex. if you know $-2-\sqrt{3}$ is a solution, then $-2+\sqrt{3}$ is a solution

Ex. if you know 5 is a solution, no other solutions can be assumed – no radical or imaginary component

Think about the quadratic equation – this equation suggest these theorems $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Ex. Write a polynomial function *f* of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 2 and 3 + *i*.

$f(x) = (x-2)[x-(3+i)][x-(3-i)]$ Write $f(x)$ in factored form.

$= (x-2)[(x-3)-i][(x-3)+i]$ Regroup terms.

$= (x-2)[(x-3)^2 - i^2]$ Multiply.

$= (x-2)[(x^2-6x+9)-(-1)]$ Expand binomial and use $i^2 = -1$.

$= (x-2)(x^2-6x+10)$ Simplify.

$= x^3 - 6x^2 + 10x - 2x^2 + 12x - 20$ Multiply.

$= x^3 - 8x^2 + 22x - 20$ Combine like terms.

Modelling data to polynomial equation

Given intercepts/zeroes (ex. 2, 3, -1, -5, etc.) + 1 other point [ex. (6,9)]

1) y = a(x-2)(x-3)(x+1)(x+5)

2) Plug other point into equation to solve for "a" the leading coefficient

9 = a(6-2)(6-3)(6+1)(6+5)

3) Write the equation

Finite differences

If given n^{th} order polynomial, then the n^{th} finite difference of equally spaced data will be constant.