Example 5.4.3

If $\mu = E(Y)$ is the parameter being estimated, then the sample mean $\bar{Y} = (1/n) \cdot \sum_{i=1}^{n} Y_i$ is always an unbiased estimator. This is an immediate consequence of the distributive property of expected values:

$$E(\bar{Y}) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{n\mu}{n} = \mu$$

Example 5.4.4

The statistic

$$\frac{1}{n} \sum_{i=1}^{n} (Y_i - \mu)^2$$

is an unbiased estimator for σ^2 (see Question 5.4.10). In many cases, though, we must estimate σ^2 without knowing μ. The statistic often used in that situation is the *sample variance* S^2, where

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2$$

Dividing the sum of the squared deviations by $n - 1$ (rather than n) is necessary to make S^2 unbiased. We can write

$$E\left[\sum_{i=1}^{n} (Y_i - \bar{Y})^2\right] = E\left[\sum_{i=1}^{n} (Y_i^2 - 2Y_i\bar{Y} + \bar{Y}^2)\right]$$

$$= E\left[\left(\sum_{i=1}^{n} Y_i^2 - n\bar{Y}^2\right)\right]$$

$$= \sum_{i=1}^{n} E(Y_i^2) - nE(\bar{Y}^2)$$

But

$$E(Y_i^2) = \sigma^2 + \mu^2$$

and

$$E(\bar{Y}^2) = \frac{\sigma^2}{n} + \mu^2 \qquad \text{(why?)}$$

Therefore,

$$E\left[\sum_{i=1}^{n} (Y_i - \bar{Y})^2\right] = (n\sigma^2 + n\mu^2) - \sigma^2 - n\mu^2 = (n-1)\sigma^2$$

so

$$E(S^2) = \frac{1}{n-1}(n-1)\sigma^2 = \sigma^2$$